Arctan(ktan()) Modified

Modification

The function $f(x) = \arctan(k \tan x)$ is undefined at the zeros of cosine: $Z = \{x | \cos(x) = 0\} = \left\{..., -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, ...\right\}$, and is inconsistent near these values in the sense that the limits from the left and right do not agree. For example:

$$\lim_{x \to \frac{\pi}{2}^-} = \frac{\pi}{2} \quad \text{but} \quad \lim_{x \to \frac{\pi}{2}^+} = -\frac{\pi}{2}.$$

The formula for the derivative however makes sense on the whole real line.

$$f'(x) = \frac{k}{\cos^2 x + k^2 \sin^2 x}.$$

This suggests defining

$$f^\star(x) = \int_0^x \frac{k}{\cos^2 t + k^2 \sin^2 t} dt.$$

The function $f^\star$ agrees with f on $(-\frac{\pi}{2}, \frac{\pi}{2})$. For $f^\star$ to be continuous and have the correct derivative where f is defined, it must be

$$f^\star(x) = \begin{cases} x & k > 0, x \in Z \\ \arctan(k \tan x) + n\pi & k > 0, x \notin Z \\ -x & k < 0, x \in Z \\ \arctan(k \tan x) - n\pi & k < 0, x \notin Z \end{cases}$$

where $n = \text{round}(\frac{x}{\pi})$.

While mathematically correct, this formula is awkward to implement since the condition $x \in Z$ must be tested using a tolerance. Instead, notice that the function

$$g(x) = \text{atan2}(k \sin x, \cos x)$$

is equal to $f^\star$ on the interval $(-\pi, \pi)$, (by checking $g(0)$ and the derivative). The graph of $f^\star$ may be seen as composed of diagonal translates of the graph of g restricted to the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$.

$$f^\star(x) = \begin{cases} g(x - n\pi) + n\pi & k > 0 \\ g(x - n\pi) - n\pi & k < 0 \end{cases}$$

where again $n = \text{round}(\frac{x}{\pi})$.

Stability

Even with floating point error, g is only evaluated within or near the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ where it is highly stable and agrees with $f^\star$. Also, potentially ambiguous values of x are handled correctly. For example, suppose $k > 0$ and $x = \langle\frac{3\pi}{2}\rangle$ where angle brackets mean approximately. Then $n = \text{round}(\langle\frac{3}{2}\rangle)$, which might be either 1 or 2, but it doesn't matter.

$$g(x - 1\pi) + 1\pi = g(\langle\frac{\pi}{2}\rangle) + \pi = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$$
$$g(x - 2\pi) + 2\pi = g(\langle-\frac{\pi}{2}\rangle) + 2\pi = -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}$$

Graphs

The modified function $f^\star$ is shown for different values of k. The curve within the center open square agrees with the original f while the other squares enclose the diagonal translates.