Animating Planetary Motion

Animating planetary orbits is easier than understanding how these orbits result from a central force. In this document, the animation formulae are presented first, but in such a way that the central force may be found by continuing the calculation. The opposite approach is often taken. See for example

https://physics.stackexchange.com/questions/353239/elliptic-orbit-solution-based-on-initial-conditions.

Circular Orbits

Let r be the radius and let T be the period. Then the animation is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} r\cos\frac{2\pi t}{T} \\ r\sin\frac{2\pi t}{T} \end{bmatrix}$$

where t is the elapsed time. Continuing the calculation,

$$\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} = -\left(\frac{2\pi}{T}\right)^2 \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

showing that the acceleration is central. The force law is apparently linear, but since r is constant, the acceleration is also consistent with the inverse square law. If the right side is compared with $\frac{-GM}{r^3}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, it is found that $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$.

Eliptical Orbits

Let a, b and c be the semi-axes and distance to foci of the ellipse, where $a^2 = b^2 + c^2$. Parametrize the ellipse so that one focus is at the origin:

$$\begin{aligned} x(\alpha) &= a\cos\alpha + c \\ y(\alpha) &= b\sin\alpha \end{aligned}$$

where $\alpha \in [0, 2\pi]$. Note that α is not the polar angle of the point $(x(\alpha), y(\alpha))$, but is the angle from the center of the ellipse.

Swept Area

Let $A(\alpha)$ be the area swept out. The rate of change of area with respect to α is:

$$A'(\alpha) = \frac{1}{2}\,\text{det}\left[\begin{array}{c|c} x(\alpha) & x'(\alpha) \\ y(\alpha) & y'(\alpha) \end{array}\right]$$
$$= \frac{1}{2}\left(ab + bc\cos\alpha\right)$$

Integrate arranging that $A(0) = 0$.

$$A(\alpha) = \frac{1}{2}\left(ab\alpha + bc\sin\alpha\right).$$

Notice for example that $A(\pi/2)$ is the sum of the area of a triangle and a quarter the area of the ellipse.

Inverse of the Area Function

It will be necessary to calculate $\alpha = A^{-1}(S)$ where S is a given amount of swept area. To this end, let $[p_i, q_i]$ be a sequence of brackets for α, using the fact that $A(\alpha)$ is an increasing function. To find the inverse for any S, let

$$[p_0, q_0] = \left[2\pi\text{floor}\left(\frac{S}{\pi ab}\right), 2\pi\text{ceil}\left(\frac{S}{\pi ab}\right)\right]$$

which will simply be $[0, 2\pi]$ when $S \in (0, \pi ab)$. Then, according to the bisection method, iterate

$$\alpha_i = \frac{p_i + q_i}{2}$$

$$[p_{i+1}, q_{i+1}] = \begin{cases} [p_i, \alpha_i] & \text{if } A(\alpha_i) > S \\ [\alpha_i, q_i] & \text{if } A(\alpha_i) < S \end{cases}$$

so that $\alpha_i \to \alpha$ until $q_i - p_i$ is smaller than the desired tolerance.

Parametrization by Time

For elapsed time t and period T, the fraction of the time elapsed must equal the fraction of the area swept.

$$\frac{t}{T} = \frac{A(\alpha)}{\pi ab}$$

Define

$$f(t) = \alpha = A^{-1}\left(\frac{\pi ab}{T}t\right)$$

The animation is then given by

$$\begin{aligned} x(t) &= a \cos f(t) + c \\ y(t) &= b \sin f(t). \end{aligned}$$

The Changing Radius

Let $r(t)$ be the distance from the origin to the point $(x(t), y(t))$ and use the relation $b^2 = a^2 - c^2$.

$$\begin{aligned} (r(t))^2 &= (x(t))^2 + (y(t))^2 \\ &= (a \cos f(t) + c)^2 + b^2 \sin^2 f(t) \\ &= a^2 \cos^2 f(t) + 2ac \cos f(t) + c^2 + \left(a^2 - c^2\right) \sin^2 f(t) \\ &= (a + c \cos f(t))^2 . \end{aligned}$$

Therefore $r(t) = a + c \cos f(t)$.

Derivatives of f

Write

$$\frac{1}{2} \left(abf(t) + bc \sin f(t)\right) = A(f(t)) = \frac{\pi ab}{T} t$$

and differentiate implicitly.

$$f'(t) = \frac{2\pi}{T} \frac{1}{(1 + \frac{c}{a} \cos f(t))}.$$

The second deriviative is then

$$f''(t) = \left(\frac{2\pi}{T}\right)^2 \frac{\frac{c}{a} \sin f(t)}{(1 + \frac{c}{a} \cos f(t))^3}.$$

Velocity

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -a \sin f(t) \\ b \cos f(t) \end{bmatrix} f'(t).$$

Acceleration

$$\begin{aligned}
\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} &= \begin{bmatrix} -a \sin f(t) \\ b \cos f(t) \end{bmatrix} f''(t) + \begin{bmatrix} -a \cos f(t) \\ -b \sin f(t) \end{bmatrix} (f'(t))^2 \\
&= -\left(\frac{2\pi}{T}\right)^2 \frac{1}{(1+\frac{c}{a}\cos f(t))^3} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \\
&= -\left(\frac{2\pi}{T}\right)^2 \frac{a^3}{(r(t))^3} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.
\end{aligned}$$

If this is compared with $\frac{-GM}{(r(t))^3} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ it is found that $\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$

For example:
$M_{\text{sun}} = 1.98847 \times 10^{30}$kg
$G = 6.67430 \times 10^{-11}$Nm^2/kg^2
$a_{\text{earth}} = 149.60 \times 10^9$m
$T_{\text{earth}} = 31558150$ seconds

Initial Conditions

$$\begin{aligned}
f(0) &= 0 \\
f'(0) &= \frac{2\pi}{T}\frac{1}{1+\frac{c}{a}}
\end{aligned}$$

$$\begin{aligned}
x(0) &= a + c \\
y(0) &= 0 \\
x'(0) &= 0 \\
y'(0) &= b\frac{2\pi}{T}\frac{1}{1+\frac{c}{a}}
\end{aligned}$$

Notes

Inverting the area formula is Kepler's anomaly method in disguise, but the area formula is more general. For any parametrized curve $(x(\alpha), y(\alpha))$ that is oriented counterclockwise relative to the origin, the area swept out satisfies

$$A'(\alpha) = \frac{1}{2} \det \left[\begin{array}{c|c} x(\alpha) & x'(\alpha) \\ y(\alpha) & y'(\alpha) \end{array}\right].$$

This may be proven directly or as a special case of Green's theorem. It may be also be shown that $A'(\alpha)$ is constant if and only if $(x(\alpha), y(\alpha))$ is parallel to $(x''(\alpha), y''(\alpha))$. In the special case when the parameter is time,

$$A'(t) = \frac{1}{2}\,\mathsf{det}\left[\begin{array}{c|c} x(t) & x'(t) \\ y(t) & y'(t) \end{array}\right]$$

so that sweeping equal areas in equal times is equivalent to a central acceleration. When mass is considered and $z(t) \equiv 0$, the angular momemtum is

$$\mathbf{L} = m\left(0, 0, \mathsf{det}\left[\begin{array}{c|c} x(t) & x'(t) \\ y(t) & y'(t) \end{array}\right]\right) = (0, 0, 2mA'(t))$$

In the case of the elliptical orbit, using the formulas for position and velocity, the derivative of area with respect to time is

$$\begin{aligned} A'(t) &= \frac{1}{2}\,\mathsf{det}\left[\begin{array}{c|c} a\cos f(t) + c & -a\sin f(t) \\ b\sin f(t) & b\cos f(t) \end{array}\right]\frac{2\pi}{T}\frac{1}{(1 + \frac{c}{a}\cos f(t))} \\ &= \frac{\pi ab}{T} \end{aligned}$$

as expected.

Elliptical Orbits with Precession

In keeping with the theme of starting with the animation, and later finding the force, seek a solution of the form

$$\left[\begin{array}{c} X(t) \\ Y(t) \end{array}\right] = \left[\begin{array}{c|c} \cos g(t) & -\sin g(t) \\ \sin g(t) & \cos g(t) \end{array}\right]\left[\begin{array}{c} x(t) \\ y(t) \end{array}\right],$$

where the previously found solution $(x(t), y(t))$ is rotated by the angle $g(t)$. Then

$$\left[\begin{array}{c} X'(t) \\ Y'(t) \end{array}\right] = \left[\begin{array}{c|c} \cos g(t) & -\sin g(t) \\ \sin g(t) & \cos g(t) \end{array}\right]\left[\begin{array}{c} x'(t) - y(t)g'(t) \\ y'(t) + x(t)g'(t) \end{array}\right]$$

The swept area is given by

$$\begin{aligned} A'(t) &= \frac{1}{2}\,\mathsf{det}\left[\begin{array}{c|c} X(t) & X'(t) \\ Y(t) & Y'(t) \end{array}\right] \\ &= \frac{\pi ab}{T} + \frac{1}{2}((x(t))^2 + (y(t))^2)g'(t). \end{aligned}$$

In order to sweep equal areas in equal times, let H be a constant such that

$$g'(t) = \frac{H}{(x(t))^2 + (y(t))^2}$$

Recalling that $r(t) = a + c\cos f(t)$, write

$$g'(t) = \frac{H}{(a + c\cos f(t))^2}$$

To find $g(t)$, let $I = \int \frac{1}{(a+c\cos f(t))^2} dt$. Substitute $u = f(t)$ to obtain

$$I = \frac{T}{2\pi a} \int \frac{1}{a + c\cos u} du$$

Then using Weierstrass substitution $w = \tan \frac{u}{2}$,

$$I = \frac{T}{2\pi a} \int \frac{1}{a + c\frac{1-w^2}{1+w^2}} \frac{2}{1 + w^2} dw$$
$$= \frac{T}{\pi a\sqrt{(a+c)(a-c)}} \arctan\left(\sqrt{\frac{a-c}{a+c}} w\right)$$

Replace $\arctan(k\tan())$ with $\text{arctanktan}\,()$, the unique function that agrees with $\arctan(k\tan())$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and has derivative $\frac{k}{\cos^2 x + k^2 \sin^2 x}$ on the whole of $\mathbb{R}$. Here $k = \sqrt{\frac{a-c}{a+c}}$. This gives the formula for $g(t)$.

$$g(t) = H\frac{T}{\pi ab} \text{arctanktan}\left(\frac{f(t)}{2}\right)$$

It remains to interpret the constant H. The function f maps whole multiples of the period T to whole multiples of 2π, and the function $\text{arctanktan}\,()$ acts as the identity on whole multiples of π.

$$f(nT) = 2\pi n$$

$$g(nT) = H\frac{T}{\pi ab} \text{arctanktan}\,(\pi n)$$
$$= H\frac{T}{\pi ab} \pi n$$

On the other hand, if the solution $(X(t), Y(t))$ has period P, then $g(P) = 2\pi$. This permits solving for H in the case that P is a whole multiple of T.

$$H = \frac{2\pi ab}{P}$$

This completes the animation formula, where P is the (rather large) period of precession, a more intuitive input parameter than H.

By writing H in terms of P, the rate of change of area becomes

$$A'(t) = \frac{\pi ab}{T} + \frac{\pi ab}{P}.$$

If the animation without precession were run for total time nT, the area would be traced out n times, but at the rate above, it is traced one more time, suggesting that $n + 1$ is the winding number of the path.

Acceleration

To make the calculation less cluttered, drop the (t). Differentiating the velocity, the acceleration may be written

$$\left[\begin{array}{c} X'' \\ Y'' \end{array}\right] = \left[\begin{array}{c|c} \cos g & -\sin g \\ \sin g & \cos g \end{array}\right] \left\{ \left[\begin{array}{c} x'' \\ y'' \end{array}\right] + \left[\begin{array}{c} E \\ F \end{array}\right] \right\}$$

where

$$\begin{aligned} E &= -yg'' - 2y'g' - x(g')^2 \\ F &= xg'' + 2x'g' - y(g')^2. \end{aligned}$$

Now recall that g' and hence g'' may be written in terms of r.

$$g' = \frac{H}{x^2 + y^2} = \frac{H}{r^2}$$

$$g'' = \frac{-2H(xx' + yy')}{(x^2 + y^2)^2} = \frac{-2H(xx' + yy')}{r^4}.$$

This leads to

$$\begin{aligned} E &= \frac{1}{r^4} Hx \left(-4\frac{\pi ab}{T} - H \right) \\ F &= \frac{1}{r^4} Hy \left(-4\frac{\pi ab}{T} - H \right). \end{aligned}$$

It follows that

$$\left[\begin{array}{c} X''(t) \\ Y''(t) \end{array}\right] = -\left(\left(\frac{2\pi}{T} \right)^2 \frac{a^3}{(r(t))^3} + \frac{1}{(r(t))^4} H \left(4\frac{\pi ab}{T} + H \right) \right) \left[\begin{array}{c} X(t) \\ Y(t) \end{array}\right].$$

The magnitude of the central force has the form inverse square plus inverse cube.

Initial Conditions

$$\begin{aligned} g(0) &= 0 \\ g'(0) &= \frac{H}{(a + c)^2} \end{aligned}$$

$$\begin{aligned} X(0) &= x(0) \\ Y(0) &= y(0) \\ X'(0) &= 0 \\ Y'(0) &= y'(0) + x(0)g'(0) \end{aligned}$$

Sun Centered Elliptical Orbits

It is perhaps worth mentioning the animation

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a \cos \frac{2\pi t}{T} \\ b \sin \frac{2\pi t}{T} \end{bmatrix}$$

which does sweep equal areas in equal times.
It follows that

$$\begin{bmatrix} x''(t) \\ y''(t) \end{bmatrix} = -\left(\frac{2\pi}{T}\right)^2 \begin{bmatrix} a \cos \frac{2\pi t}{T} \\ b \sin \frac{2\pi t}{T} \end{bmatrix}$$

showing that the acceleration is central and the force law is linear.